UNIKASSEL VERSITÄT

FB 10 – Institut für Mathematik
AG Algorithmische Algebra
und Diskrete Mathematik
Prof. Dr. Wolfram Koepf

INSTITUTSKOLLOQUIUM

Representation Theory of Reflection Groups

Referent:	**Dr. Hery Randriamaro (Madagaskar)**
Termin:	Montag, 24. Januar 2022, 17:15 Uhr
Ort:	Raum 1409, Heinrich-Plett-Str. 40, AVZ, Kassel-Oberzwehren

Abstract:

It all started in 1896, when Dedekind sent a letter to Frobenius. He wrote about the group determinant, explained how it factors in the abelian case, and suggested him to think about the nonabelian case. It is the question of factoring the group determinant of an arbitrary group that gave rise to representation theory by Frobenius. This latter initially developed the representation of groups essentially based on the character groups. But over time, the theory has provided groups with concrete descriptions in terms of linear algebra. Namely, each element of a group is represented by a matrix in such a way that the group operation is matrix multiplication. The most studied representation until now is that of reflection groups. A reflection is a map on a structured object preserving its structure. These groups arise in a multitude of ways in mathematics. Moreover, many groups are isomorphic to some reflection groups. Classical examples are the Coxeter groups, the symmetry groups of regular polytopes, and the symmetric groups. These latter are particularly important as the Cayley theorem states that every group is isomorphic to a subgroup of a symmetric group. And reflection groups may be regarded as the foundation of other algebraic structures. Like the descent algebras which are subalgebras of the group algebras of reflection groups, and the Hecke algebras which are deformations of the group algebras of reflection groups. Representation theory naturally extends to representation theory of algebras since every group can be extended to group algebras. The presentation is essentially divided into three parts: the origin of representation theory, representation theory of reflection groups and related algebras, and applications of representations of groups and related algebras. Besides, it is good to warn the experts on the subject that the presentation is aimed to be instructive, in the sense that enough time is planned to be spent on basic definitions.

gez. Prof. Dr. Wolfram Koepf

Representation Theory of Reflection Groups

Hery Randriamaro

Universität Kassel

Institutskolloquium - January 24, 2022

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Origin

Take n variables $x_0, \ldots, x_{n-1}$. Catalan introduced in 1846 the **circulant** of order n

$$c_n = \begin{vmatrix} x_0 & x_{n-1} & \cdots & x_1 \\ x_1 & x_0 & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_{n-2} & \cdots & x_0 \end{vmatrix}.$$

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For a finite group G, form a set of variables $\{x_g\}_{g \in G}$. As generalization of the circulant, Dedekind considered the **group determinant**

$$\Theta(G) := \left| x_{gh^{-1}} \right|_{g,h \in G}.$$

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Origin

The **character group** $\hat{G}$ of an abelian group G is the group of homomorphisms from G to $\mathbb{C}^*$. Dedekind proved in 1880 that

$$\Theta(G) = \prod_{\chi \in \hat{G}} \Big(\sum_{g \in G} \chi(g) x_g \Big).$$

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The circulant of order n is $c_n = \Theta(\mathbb{Z}/n\mathbb{Z})$, and $\widehat{\mathbb{Z}/n\mathbb{Z}}$ is composed by the homomorphisms

$$h_k : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*,\ \bar{j} \mapsto \exp \frac{2jk\pi i}{n} \quad \text{with} \quad k \in \{0, 1, \dots, n-1\}.$$

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Therefore

$$c_n = \prod_{k=0}^{n-1} \sum_{j=0}^{n-1} x_i \exp \frac{2jk\pi i}{n}.$$

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Origin

Then, he became interested in factoring $\Theta(G)$, if G is a nonabelian finite group. He discovered that, when G is nonabelian, some of the irreducible factors of $\Theta(G)$ may be nonlinear.

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Let us enumerate the elements of $\mathfrak{S}_3$ as Dedekind did:

$$\mathbf{1} = (1),\ \mathbf{2} = (123),\ \mathbf{3} = (132),\ \mathbf{4} = (23),\ \mathbf{5} = (13),\ \mathbf{6} = (12).$$

He obtained $\Theta(\mathfrak{S}_3) = \Phi_1 \Phi_2 \Phi_3^2$ with

$$\begin{aligned}\Phi_1 &= x_{\mathbf{1}} + x_{\mathbf{2}} + x_{\mathbf{3}} + x_{\mathbf{4}} + x_{\mathbf{5}} + x_{\mathbf{6}},\\ \Phi_2 &= x_{\mathbf{1}} + x_{\mathbf{2}} + x_{\mathbf{3}} - x_{\mathbf{4}} - x_{\mathbf{5}} - x_{\mathbf{6}},\\ \Phi_3 &= x_{\mathbf{1}}^2 + x_{\mathbf{2}}^2 + x_{\mathbf{3}}^2 - x_{\mathbf{4}}^2 - x_{\mathbf{5}}^2 - x_{\mathbf{6}}^2 - x_{\mathbf{1}}x_{\mathbf{2}} - x_{\mathbf{1}}x_{\mathbf{3}} - x_{\mathbf{2}}x_{\mathbf{3}}\\ &\quad + x_{\mathbf{4}}x_{\mathbf{5}} + x_{\mathbf{4}}x_{\mathbf{6}} + x_{\mathbf{5}}x_{\mathbf{6}}.\end{aligned}$$

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Origin

In 1896, Dedekind wrote a letter to Frobenius in which he mentioned the group determinant $\Theta(G)$, explained how it factors in the abelian case, and suggested him to think about the nonabelian case. *It is the question of factoring $\Theta(G)$ for an arbitrary finite group G that gave rise to representation theory by Frobenius.*

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Representation of Groups

The theory provides groups with concrete descriptions in terms of linear algebra. Let G be a group, V a vector space over a field $\mathbb{K}$ of characteristic 0, and $\mathrm{GL}(V)$ the general linear group on V. A **representation** of G on V is a group homomorphism

$$\mathscr{X}_V : G \to \mathrm{GL}(V),$$

and the vector space V is called a G-**module**.

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Representation of Groups

The action of $\mathfrak{S}_n$ on $V = \{c_1\mathbf{1} + \cdots + c_n\mathbf{n} \mid c_1, \dots, c_n \in \mathbb{R}\}$ defined, for every $\sigma \in \mathfrak{S}_n$, by

$$\sigma(c_1\mathbf{1} + \cdots + c_n\mathbf{n}) = c_1\sigma(\mathbf{1}) + \cdots + c_n\sigma(\mathbf{n})$$

is a representation. We use the basis $\{\mathbf{1}, \dots, \mathbf{n}\}$ to compute the images of the representation.

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is a representation. We use the basis $\{\mathbf{1}, \ldots, \mathbf{n}\}$ to compute the images of the representation. For $\sigma = (12)$ for example, we have

$$\sigma(\mathbf{1}) = \mathbf{2}, \quad \sigma(\mathbf{2}) = \mathbf{1}, \quad \sigma(\mathbf{k}) = \mathbf{k} \text{ if } \mathbf{k} \in \{\mathbf{3}, \ldots, \mathbf{n}\},$$

and so

$$\mathscr{X}_V(\sigma) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

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Representation of Algebras

In the same way that a group G can be extended to a group algebra

$$\mathbb{K}[G] = \Big(\sum_{g \in G} x_g g \mid x_g \in \mathbb{K} \Big),$$

a group representation can be extended to an algebra representation.

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Representation of Algebras

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$$\mathbb{K}[G] = \Big(\sum_{g \in G} x_g g \mid x_g \in \mathbb{K} \Big),$$

a group representation can be extended to an algebra representation.

Let A be a finitely generated algebra and V a vector space, both over a field $\mathbb{K}$ of characteristic 0, and $\mathrm{End}(V)$ the algebra of linear transformations in V. A **representation** of A on V is an algebra homomorphism

$$\mathscr{X}_V : A \to \mathrm{End}(V).$$

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Representation of Algebras

Consider the action of $\mathfrak{S}_n$ on $V = \{c_1\mathbf{1} + \cdots + c_n\mathbf{n} \mid c_1, \ldots, c_n \in \mathbb{R}\}$, and the permutation $\tau = (123)$. We have

$$\tau(\mathbf{1}) = \mathbf{2}, \quad \tau(\mathbf{2}) = \mathbf{3}, \quad \tau(\mathbf{3}) = \mathbf{1}, \quad \tau(\mathbf{k}) = \mathbf{k} \text{ if } \mathbf{k} \in \{\mathbf{4}, \ldots, \mathbf{n}\}.$$

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So, for $a, b \in \mathbb{C}$, we have

$$\mathscr{X}_V(a\sigma + b\tau) = a\mathscr{X}_V(\sigma) + b\mathscr{X}_V(\tau) = \begin{pmatrix} 0 & a+b & 0 & \cdots & 0 \\ a & 0 & b & \cdots & 0 \\ b & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

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Representation of Algebras

The most important representations of a group G and its group algebra $\mathbb{K}[G]$ is the **regular representation**. It represents the action of G or $\mathbb{K}[G]$ by left multiplication on $\mathbb{K}[G]$ itself. A basis of $\mathbb{K}[G]$ as G-module is G.

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The group determinant of a group G defined by Dedekind is the determinant of the regular representation of the element $\sum_{g \in G} x_g g \in \mathbb{K}[G]$.

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The group determinant of a group G defined by Dedekind is the determinant of the regular representation of the element $\sum_{g \in G} x_g g \in \mathbb{K}[G]$.

In other words,

$$\Theta(G) = \det \mathscr{X}_{\mathbb{K}[G]}\Big(\sum_{g \in G} x_g g\Big).$$

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Representation Reduction

Let G be a finite group, and V a nonzero G-module. A representation on V is said to be **irreducible** if V contains no subspace U such that $\{0\} \subsetneq U \subsetneq V$ and U is a G-module.

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The theorem of Maschke states that

$$V = \bigoplus_{i=1}^{k} V_i,$$

where each subspace V_i of V is an irreducible G-module.

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$$V = \bigoplus_{i=1}^{k} V_i,$$

where each subspace V_i of V is an irreducible G-module.

There is a fixed $\dim V \times \dim V$ matrix T such that, for every $g \in G$,

$$T \mathscr{X}_V(g) T^{-1} = \bigoplus_{i=1}^{k} \mathscr{X}_{V_i}(g).$$

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Representation Reduction

Let G be a finite group, and denote $\mathrm{Cl}(G)$ its set of conjugacy classes:

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Representation Reduction

Let G be a finite group, and denote $\mathrm{Cl}(G)$ its set of conjugacy classes:

- G has $\#\mathrm{Cl}(G)$ different irreducible representations on G-modules V_i up to isomorphism such that

$$\mathbb{K}[G] = \bigoplus_{i=1}^{\#\mathrm{Cl}(G)} m_i V_i \quad \text{with} \quad m_i V_i = \overbrace{V_i \oplus \cdots \oplus V_i}^{m_i \text{ times}},$$

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- $m_i = \dim V_i$ and $\displaystyle\sum_{i=1}^{\#\mathrm{Cl}(G)} (\dim V_i)^2 = \#G$,
- there is a fixed $\#G \times \#G$ matrix T such that, for all $a \in \mathbb{K}[G]$,

$$T\,\mathscr{X}_{\mathbb{K}[G]}(a)\,T^{-1} = \bigoplus_{i=1}^{\#\mathrm{Cl}(G)} \dim V_i\ \mathscr{X}_{V_i}(a).$$

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Real Reflection Groups

Many groups are isomorphic to some reflection groups:

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- the symmetry groups of regular polytopes,

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- the braid groups in knot theory,

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The theorem of Cayley states that every finite group is isomorphic to a subgroup of a symmetric group.

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Reflection groups are the foundation of other algebraic structures such as descent algebras and Hecke algebras.

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Real Reflection Groups

Endow $\mathbb{R}^n$ with the usual unitary inner product $\langle \centerdot, \centerdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The **reflection** $s_u : \mathbb{R}^n \to \mathbb{R}^n$ across the hyperplane $u^\perp$ in $\mathbb{R}^n$ is defined by

$$s_u(x) := x - 2\frac{\langle x, u\rangle}{\langle u, u\rangle} u.$$

A **reflection group** is a finite group generated by reflections in $\mathbb{R}^n$.

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A **reflection group** is a finite group generated by reflections in $\mathbb{R}^n$.

Let W be a reflection group. One can fix a minimal set S of reflections in W such that $\langle S \rangle = W$. The pair (W, S) is called a **Coxeter system**.

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The matrix $\mathrm{M}_W = (m_{st})_{s,t \in S}$ such that m_{st} is the order of $st \in W$ is the **Coxeter matrix** of W.

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Real Reflection Groups

Let W be a reflection group, and V a nontrivial subspace of $\mathbb{R}^n$ such that $w(V) \subseteq V$ for every $w \in W$. The reflection group W is said to be **irreducible** on V if V contains no subspace U such that

$$\{0\} \subsetneq U \subsetneq V \quad \text{and} \quad \forall w \in W,\, w(U) \subseteq U.$$

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Coxeter proved in 1934 that, up to isomorphism, the irreducible reflection groups are

- the classical infinite reflection groups A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$,

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- the exceptional reflection groups E_6, E_7, E_8, F_4, G_2,

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Coxeter proved in 1934 that, up to isomorphism, the irreducible reflection groups are

- the classical infinite reflection groups A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$,
- the exceptional reflection groups E_6, E_7, E_8, F_4, G_2,
- the non-crystallographic reflection groups H_3, H_4, $I_2(m)$ $(m \in \mathbb{N} \setminus \{1, 2, 3, 4, 6\})$.

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Specht Module

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$. The **Ferrers diagram** of shape λ is an array of n dots having l left-justified rows with row i containing λ_i dots.

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The partition $(3, 1)$ of 4 has Ferrers diagram
$$\begin{matrix} \bullet & \bullet & \bullet \\ \bullet & & \end{matrix}.$$

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The partition $(3, 1)$ of 4 has Ferrers diagram
$$\begin{matrix} \bullet & \bullet & \bullet \\ \bullet & & \end{matrix}.$$

A **Young tableau** of shape λ is an array obtained by replacing the dots of the Ferrers diagram of shape λ with the numbers $1, 2, \ldots, n$ bijectively.

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The partition $(3, 1)$ of 4 has Ferrers diagram $\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & & \end{array}$.

A **Young tableau** of shape λ is an array obtained by replacing the dots of the Ferrers diagram of shape λ with the numbers $1, 2, \ldots, n$ bijectively.

Some Young tableaux of shape $(3, 1)$ are $\begin{array}{lll} 1 & 2 & 3 \\ 4 & & \end{array}$, $\begin{array}{lll} 3 & 2 & 1 \\ 4 & & \end{array}$, $\begin{array}{lll} 2 & 1 & 4 \\ 3 & & \end{array}$.

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Specht Module

Two tableaux of shape λ are row equivalent if their corresponding rows contain the same elements. A **tabloid** is a class of the row equivalence.

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Two tableaux of shape λ are row equivalent if their corresponding rows contain the same elements. A **tabloid** is a class of the row equivalence.

The class $\begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \cline{1-1} \end{array} =$

$$\left\{ \begin{matrix} 1 & 2 & 3 \\ 4 & & \end{matrix}, \begin{matrix} 1 & 3 & 2 \\ 4 & & \end{matrix}, \begin{matrix} 2 & 1 & 3 \\ 4 & & \end{matrix}, \begin{matrix} 2 & 3 & 1 \\ 4 & & \end{matrix}, \begin{matrix} 3 & 1 & 2 \\ 4 & & \end{matrix}, \begin{matrix} 3 & 2 & 1 \\ 4 & & \end{matrix} \right\}$$

is a tabloid of shape λ.

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Specht Module

Suppose that the tableau t has columns $C_1, C_2, \ldots, C_k$, and let

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Specht Module

Suppose that the tableau t has columns $C_1, C_2, \ldots, C_k$, and let

$$C_t = \mathfrak{S}_{C_1} \times \mathfrak{S}_{C_2} \times \ldots \mathfrak{S}_{C_k},$$
$$\kappa_t = \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma)\sigma.$$

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The **polytabloid** associated to t is $e_t = \kappa_t \bar{t}$.

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The **polytabloid** associated to t is $e_t = \kappa_t \bar{t}$.

If $t = \begin{matrix} 1 & 2 & 3 \\ 4 & & \end{matrix}$, then $\kappa_t = (1) - (12)$ and

$$e_t = \begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \cline{1-1} \end{array} - \begin{array}{ccc} \hline 4 & 2 & 3 \\ \hline 1 & & \\ \cline{1-1} \end{array}.$$

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Specht Module

The **Specht module** S^λ corresponding to a partition $\lambda \vdash n$ is the $\mathfrak{S}_n$-module spanned by the polytabloids e_t, where t is of shape λ.

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Specht Module

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A Young tableau is **standard** if rows and columns are increasing sequences. A Specht module is irreducible, and the set $\{e_t \mid t \text{ is a standard tableau of shape } \lambda\}$ is a basis for S^λ.

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If f^λ is the number of standard tableaux of shape λ, Specht proved in 1935 that $\dim S^\lambda = f^\lambda$, and

$$\mathbb{R}\mathfrak{S}_n \cong \bigoplus_{\lambda \vdash n} f^\lambda S^\lambda.$$

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Double Coset Representatives

Consider a Coxeter system (W, S).

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Double Coset Representatives

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For $J \subseteq S$, denote W_J the subgroup $\langle J \rangle$ of W.

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Double Coset Representatives

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For $J \subseteq S$, denote W_J the subgroup $\langle J \rangle$ of W.

For $J, K \subseteq S$, and $w \in W$, the (W_J, W_K)-**double coset** of w is the set

$$W_J w W_K := \{uwv \mid u \in W_J,\ v \in W_K\}.$$

The set of all double cosets is denoted $W_J \backslash W / W_K$.

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Let ${}^J W^K$ be a set of representatives of the double cosets in $W_J \backslash W / W_K$.
If $J = \varnothing$, we just write W^K.

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Descent Algebra

For $J \subseteq S$, let

$$x_J = \sum_{w \in W^J} w.$$

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For $J \subseteq S$, let

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For $J, K \subseteq S$, we have

$$x_J x_K = \sum_{L \subseteq K} a_{JKL} x_L$$

$$\text{with } a_{JKL} = \#\{w \in {}^J W^K \mid w^{-1} W_J w \cap W_K = W_L\}.$$

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The **descent algebra** of a Coxeter system (W, S), defined in 1976 by Solomon, is the subalgebra $\mathrm{D}_W := \mathbb{R}[x_J \mid J \subseteq S]$ of the $\mathbb{R}[W]$.

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Descent Algebra Spectrum

The total order $\succ$ on $S = \{s_i\}_{i \in [n]}$, introduced in 1992 by Bergeron and Bergeron, is defined as follows: Write $\min J = \min\{i \in [n] \mid s_i \in J\}$ and assume $\min \varnothing = n+1$. Let $J, K \subseteq S$ such that $J \neq K$:

- if $\min J > \min K$ then $J \succ K$,
- otherwise $J \succ K$ if and only if $J \setminus \{s_{\min J}\} \succ K \setminus \{s_{\min K}\}$.

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If $S = \{s_1, s_2\}$, then $\{s_2\} \succ \{s_1\} \succ \{s_1, s_2\}$.

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If $S = \{s_1, s_2\}$, then $\{s_2\} \succ \{s_1\} \succ \{s_1, s_2\}$.

Use the ordered basis $(x_{L_i})_{i \in 2^n}$ of D_W such that $L_i \succ L_j$ if $i < j$.
If $d = \sum_{i \in 2^n} \lambda_{L_i} x_{L_i} \in \mathrm{D}_W$, then $\mathscr{X}_{\mathrm{D}_W}(d)$ is an upper triangle. Hence

$$\mathrm{Sp}\, \mathscr{X}_{\mathrm{D}_W}(d) = \Big\{ \sum_{J \subseteq S} \lambda_J a_{JKK} \Big\}_{K \subseteq S}.$$

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Descent Algebra Representation

For $J = \{s_{i_1}, \dots, s_{i_p}\}$ with $i_1 < \cdots < i_p$, let $c_J = s_{i_1} \dots s_{i_p}$ and $\tilde{c}_J$ be the conjugacy class of c_J.

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Descent Algebra Representation

For $J = \{s_{i_1}, \dots, s_{i_p}\}$ with $i_1 < \dots < i_p$, let $c_J = s_{i_1} \dots s_{i_p}$ and $\tilde{c}_J$ be the conjugacy class of c_J.

We prove in 2012 that

$$\mathrm{Sp}\, \mathscr{X}_{\mathbb{R}[W]}(d) = \mathrm{Sp}\, \mathscr{X}_{\mathrm{D}_W}(d),$$

and the multiplicity of $\sum_{J \subseteq S} \lambda_J a_{JKK}$ is $\#\tilde{c}_K$.

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Group Characters

The **character** of a representation $\mathscr{X}_V$ of a group G is the function

$$\chi_V : G \to \mathbb{K}, \quad g \mapsto \operatorname{tr} \mathscr{X}_V(g).$$

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The **character table** of a group G is an array with rows indexed by the inequivalent irreducible G-modules V and columns indexed by the conjugacy classes C of G, and whose entry in row V and column C is χ_V^C.

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Group Characters

The conjugacy classes of $\mathfrak{S}_3$ are

$$C_1 = \big\{(1)\big\},\ C_2 = \big\{(12), (13), (23)\big\},\ C_3 = \big\{(123), (132)\big\}.$$

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The conjugacy classes of $\mathfrak{S}_3$ are

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The irreducible Specht modules of $\mathfrak{S}_3$ are

- $S^{(3)}$ generated by $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$,
- $S^{(1,1,1)}$ by $\displaystyle\sum_{\sigma \in \mathfrak{S}_3} \mathrm{sgn}(\sigma)\sigma \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$,
- $S^{(2,1)}$ by $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$ - $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$ - $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$.

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The character table of $\mathfrak{S}_3$ is

	C_1	C_2	C_3
$S^{(3)}$	1	1	1
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The **character** of a representation $\mathscr{X}_V$ of an algebra A over $\mathbb{K}$ is the function

$$\chi_V : A \to \mathbb{K}, \quad a \mapsto \operatorname{tr} \mathscr{X}_V(a).$$

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Hecke Algebra

Consider a Coxeter system (W, S) with Coxeter matrix $(m_{st})_{s,t \in S}$.

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Consider a Coxeter system (W, S) with Coxeter matrix $(m_{st})_{s,t \in S}$.

Let $B = \{a_s, b_s \mid s \in S\}$ be a set of variables such that $a_s = a_t$ and $b_s = b_t$ whenever $s, t \in S$ are conjugate in W.

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The Hecke algebra H_W associated to W is the associative algebra over the ring $\mathbb{R}[B]$ generated by $\{T_s\}_{s\in S}$ subject to the relations

$$\forall s \in S,\ T_s^2 = a_s + b_s T_s,$$

$$\forall s, t \in S,\ s \neq t,\ \overbrace{T_s T_t T_s \dots}^{m_{st}\ times} = \overbrace{T_t T_s T_t \dots}^{m_{st}\ times}.$$

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If $w = s_1 \dots s_n$ with $s_i \in S$, define $T_w = T_{s_1} \dots T_{s_n}$.

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Skew Diagram

Let $n, k \in \mathbb{N}$ such that $n \geq k$, and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \vdash n$, $\mu = (\mu_1, \mu_2, \dots, \mu_p) \vdash k$ such that $\lambda_i \geq \mu_i$. The **skew diagram** of shape $\lambda - \mu$ is the diagram formed by the dots of the Ferrers diagram of shape λ but not that of μ.

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The skew diagram of shape $\lambda - \mu$ is a **strip** if it does not contain any 2×2 dots.

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The skew diagram of shape $\lambda - \mu$ is a **strip** if it does not contain any 2×2 dots.

Denote cc_μ^λ the number of connected components of $\lambda - \mu$.

Denote l_μ^λ the number of rows covered by $\lambda - \mu$ minus cc_μ^λ.

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Character of Hecke Algebra of Type A

Consider the Hecke algebra $\mathrm{H}_{\mathfrak{S}_n}$ with parameters $a_s = u$ and $b_s = u - 1$.

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Character of Hecke Algebra of Type A

Consider the Hecke algebra $\mathrm{H}_{\mathfrak{S}_n}$ with parameters $a_s = u$ and $b_s = u - 1$.

Let $\lambda \vdash n$, $k \in [n]$, and $w \in \mathfrak{S}_n$ such that

$$w = w'(n-k+1\; n-k+2) \dots (n-1\; n)$$

for some $w' \in \mathfrak{S}_{n-k}$. Then

$$\chi_{S^\lambda}(T_w) = \sum_{\mu \subseteq \lambda} (u-1)^{cc_\mu^\lambda - 1} (-1)^{l_\mu^\lambda} u^{k - l_\mu^\lambda - cc_\mu^\lambda} \chi_{S^\mu}(T_{w'})$$

where $\mu \vdash n-k$ such that $\lambda - \mu$ is a strip.

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Some Applications

Let G, H be groups. Formanek and Sibley proved in 1991 that

$$\Theta(G) = \Theta(H) \iff G \cong H.$$

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If two groups have different character tables, then they are not isomorphic.

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Some Applications

Let G, H be groups. Formanek and Sibley proved in 1991 that

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Jones constructed a polynomial invariant for oriented links in knot theory through characters of Hecke algebras of type A with parameters $a_s = u$ and $b_s = u - 1$ in 1987, and Geck and Lambropoulou that polynomial through characters of Hecke algebras of type B in 1997.

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Complex Reflection Groups

Endow $\mathbb{C}^n$ with the usual unitary inner product $\langle .,. \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$. A **complex reflection** $s_{u,\xi} : \mathbb{C}^n \to \mathbb{C}^n$ across the hyperplane $u^\perp$ in $\mathbb{C}^n$ is defined by $s_{u,\xi}(x) := x - (1-\xi)\dfrac{\langle x, u\rangle}{\langle u, u\rangle}u$. A **complex reflection group** is a finite group generated by complex reflections.

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Some open problems:

- irreducible representations of complex reflection groups,

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- irreducible representations of complex reflection groups,
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Some open problems:

- irreducible representations of complex reflection groups,
- existence and representation of complex descent algebras,
- existence and character of complex Hecke algebras.

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